Periodicity of Infinite Products of Matrices With Some Negative Elements and Row Sums Equal to One

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ABSTRACT

The problem of periodicity for the infinite products of a class of matrices, the **V**-matrices, with some negative elements and row sums equal to one is studied. After a series of lemmas, propositions, and theorems, a basic theorem is proved: that an infinite product of **V**-matrices under certain conditions splits into a number of subsequences which converge geometrically fast. Finally a method for finding these limits, based on some properties of generalized inverses, is provided.

1. INTRODUCTION

In Tsaklidis and Vassiliou (1990) a new class of matrices, the **V**-matrices, were introduced and their infinite products were studied. The **V**-matrices are a class of finite matrices with some negative elements and row sums equal to one. Let $\{\mathbf{V}(t) : t = 0, 1, 2 \dots\}$ be a sequence of **V**-matrices, i.e.

$$\mathbf{V}(t) = \left[\begin{array}{c|c} \mathbf{Q}(t) & \mathbf{X}(t) \\ \hline \mathbf{0} & \mathbf{Y}(t) \end{array}\right], \tag{1.1}$$

where $\{\mathbf{Q}(t) : t = 0, 1, \dots\}$ is a sequence of $k_1 \times k_1$ stochastic matrices, $\{\mathbf{Y}(t) : t = 0, 1, \dots\}$ a sequence of $k_2 \times k_2$ stochastic matrices, and $\{\mathbf{X}(t) : t = 0, 1, \dots\}$ a sequence of $k_1 \times k_2$ real matrices with the property $\mathbf{X}(t)\mathbf{1} = \mathbf{0}$,

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where **1** is the column vector of 1's and **0** the coulmn vector of 0's. Thus the **V**-matrices have row sums equal to 1, but they are not stochastic matrices, since some of the elements of the matrices $\mathbf{X}(t)$, $t = 0, 1, \ldots,$ are negative.

In Tsaklidis and Vassiliou (1990) attention was focused on infinite products of the form

$$\mathbf{V}(t, t+n) = \mathbf{V}(t)\mathbf{V}(t+1)\cdots\mathbf{V}(t+n). \tag{1.2}$$

A basic theorem was proved: that under certain conditions the limit as $n \to \infty$ of the above infinite product exists. A basic condition for the existence of the limit was—as in the case of stochastic matrices—that the sequences of matrices $\{\mathbf{Q}(t): t = 0, 1, \ldots\}$ and $\{\mathbf{Y}(t): t = 0, 1, \ldots\}$ converge to regular stochastic matrices. In the present paper attention is focused on the logical question of whether or not the $\lim_{n\to\infty}\mathbf{V}(t, t+n)$ exists if the conditions of regularity are relaxed.

In Section 2 of the present paper, after a series of lemmas, the main theorem is provided: if the sequences $\{\mathbf{Q}(t): t = 0, 1, \ldots\}$ and $\{\mathbf{Y}(t): t = 0, 1, \ldots\}$ converge to matrices $\mathbf{Q}$ and $\mathbf{Y}$ which are periodic with periods d_1 and d_2 respectively, then under certain conditions the sequence $\{\mathbf{V}(t, t+n): n = 0, 1, \ldots\}$ splits into $d_1 d_2$ subsequences for which the limits exist for every $t \in N$ if $(d_1, d_2) = 1$ (i.e. if d_1 and d_2 are relatively prime). Another important result which is proved is that if we consider each subsequence separately, then—under certain conditions—the rate of convergence of the subsequences is geometric. These results answer the important problem of periodicity for the infinite products of **V**-matrices which appear in the study of nonhomogeneous population systems as variance-covariance matrices (Vassiliou and Gerontidis, 1985; Tsaklidis and Vassiliou, 1988; Bartholomew, 1967, 1973, 1982; Pollard, 1973; Wynn, 1973).

In Section 3, through some basic lemmas, we establish a method for the calculation of the $d_1 d_2$ limits, the existence of which we proved previously.

Finally, in Section 4, we provide an illustration of the results of the present paper.

2. PERIODICITY OF INFINITE PRODUCTS OF V-MATRICES

In this section our main object is to prove that, under certain conditions, the limits of $d = d_1 d_2$ subsequences of the sequence $\{\mathbf{V}(t, t+n): n = 0, 1, \ldots\}$ exist if $(d_1, d_2) = 1$. Also it will be proved that if we consider each subsequence separately, then the rate of convergence is geometric.

Now, it is possible, for any stochastic matrix $\mathbf{A}$ of period d, to distinguish the cyclic subclasses ${}_{\mathbf{A}}C_0, {}_{\mathbf{A}}C_1, \ldots, {}_{\mathbf{A}}C_{d-1}$ [for a good account of the properties of the cyclic subclasses see Isaacson and Madsen (1976), Iosifescu (1979)]. Then the periodic stochastic matrix $\mathbf{A}$ can be written in block form. Let $\mathbf{A}_i$, $i = 0, 1, \ldots, d-1$, be the nonnull submatrices in the block form of $\mathbf{A}$ containing the transition probabilities for moving from the states of any cyclic subclass ${}_{\mathbf{A}}C_i$ to the states of the cyclic subclass ${}_{\mathbf{A}}C_{i+1}$, with ${}_{\mathbf{A}}C_d \equiv {}_{\mathbf{A}}C_0$. Consider in what follows every stochastic matrix $\mathbf{A}$ of period d to be written in block form, and symbolize by $n({}_{\mathbf{A}}C_i)$ the number of elements (states) of the subclass ${}_{\mathbf{A}}C_i$, $i = 0, 1, \ldots, d-1$.

From a combination of results in Iosifescu (1979) and Isaacson and Madsen (1976) we easily arrive at the following lemma, which describes the behavior of the powers of any periodic stochastic matrix $\mathbf{A}$.

LEMMA 2.1. *Let $\mathbf{A}$ be an irreducible stochastic matrix of period d. Then the matrix $\mathbf{A}^*(r) = \lim_{t\to\infty} \mathbf{A}^{td+r}$, $0 \leqslant r < d$, exists and is equal to $\mathbf{A}^r\mathbf{A}^*(0) = \mathbf{A}^*(0)\mathbf{A}^r$.*

REMARK 2.1. An immediate consequence of Lemma 2.1 is that if $t \in N$ is written as $nd + s$, $n, s \in N$, where $s = kd + r$, $k \in N$, $0 \leqslant r < d$, then $\mathbf{A}^*(s) = \lim_{n\to\infty} \mathbf{A}^{nd+s}$ exists and $\mathbf{A}^*(s) = \mathbf{A}^*(r)$.

REMARK 2.2. Let the $\mathbf{V}$-matrices be given by (1.1), where $\mathbf{Q}(t)$ and $\mathbf{Y}(t)$) are periodic matrices of periods d_1 and d_2 respectively. Then they can be written in block form as shown on page 178.

In what follows, the block columns of the matrix $\mathbf{X}(t)$ in the previous form of $\mathbf{V}(t)$ will be denoted by $\mathbf{X}_j(t)$, $j = 0, 1, \ldots, d_2 - 1$, i.e.

$$\mathbf{X}_j(t) = \begin{bmatrix} \mathbf{X}_{0j}(t) \\ \mathbf{X}_{1j}(t) \\ \vdots \\ \mathbf{X}_{d_1-1,j}(t) \end{bmatrix}, \qquad j = 0, 1, \ldots, d_2 - 1;$$

thus $\mathbf{X}(t)$ can be written as $\mathbf{X}(t) = (\mathbf{X}_0(t), \mathbf{X}_1(t), \ldots, \mathbf{X}_{d_2-1}(t))$. This partitioned form of $\mathbf{X}(t)$ is useful in the proof of the following lemma.

LEMMA 2.2. *Let $\mathbf{Q}$ be a $k_1 \times k_1$ stochastic matrix of period d_1, and $\mathbf{Y}$ a $k_2 \times k_2$ stochastic matrix of period d_2 with $(d_1, d_2) = 1$. Let also $\mathbf{X}(t)$ be a sequence of $k_1 \times k_2$ real matrices with $\mathbf{X}(t)\mathbf{1} = \mathbf{0}$ written in the form*

$$\mathbf{X}(t) = \big(\mathbf{X}_0(t), \mathbf{X}_1(t), \ldots, \mathbf{X}_{d_2-1}(t)\big),$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} {}_{\mathrm{Q}}C_0(t) & {}_{\mathrm{Q}}C_1(t) & \cdots & {}_{\mathrm{Q}}C_{d_1-1}(t) & {}_{\mathrm{Y}}C_0(t) & {}_{\mathrm{Y}}C_1(t) & \cdots & {}_{\mathrm{Y}}C_{d_2-1}(t) \end{array} \\
\begin{array}{c} {}_{\mathrm{Q}}C_0(t) \\ {}_{\mathrm{Q}}C_1(t) \\ \vdots \\ {}_{\mathrm{Q}}C_{d_1-1}(t) \\ {}_{\mathrm{Y}}C_0(t) \\ {}_{\mathrm{Y}}C_1(t) \\ \vdots \\ {}_{\mathrm{Y}}C_{d_2-1}(t) \end{array} &
\left[\begin{array}{cccc|cccc}
\mathbf{0} & \mathbf{Q}_0(t) & \cdots & \mathbf{0} & \mathbf{X}_{00}(t) & \mathbf{X}_{01}(t) & \cdots & \mathbf{X}_{0,\,d_2-1}(t) \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{X}_{10}(t) & \mathbf{X}_{11}(t) & \cdots & \mathbf{X}_{1,\,d_2-1}(t) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\mathbf{Q}_{d_1-1}(t) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{X}_{d_1-1,0}(t) & \mathbf{X}_{d_1-1,1}(t) & \cdots & \mathbf{X}_{d_1-1,\,d_2-1}(t) \\
\hline
 & & & & \mathbf{0} & \mathbf{Y}_0(t) & \cdots & \mathbf{0} \\
 & & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
 & \mathbf{0} & & & \vdots & \vdots & & \vdots \\
 & & & & \mathbf{Y}_{d_2-1}(t) & \mathbf{0} & \cdots & \mathbf{0}
\end{array}\right].
\end{array}
\tag{2.1}
$$

where $\mathbf{X}_j(t)$, $j = 0, 1, \ldots, d_2 - 1$, *are* $k_1 \times n({}_{\mathbf{Y}}C_j)$ *matrices. Define*

$$\mathbf{Q}^*(s) = \lim_{n\to\infty} \mathbf{Q}^{nd_1+s}, \quad \mathbf{Y}^*(t) = \lim_{n\to\infty} \mathbf{Y}^{nd_2+t}, \qquad \text{for every} \quad s, t \in N.$$

If there are column vectors $\mathbf{c}_j$ *such that*

$$\mathbf{X}_j(t)\mathbf{1} = \mathbf{c}_j \quad \text{for every } t \in N \qquad (j = 0, 1, \ldots, d_2 - 1), \tag{2.2}$$

then

$$\mathbf{B}_s = \sum_{m=0}^{d-1} \mathbf{Q}^*(f(s-m))\mathbf{X}(g(m))\mathbf{Y}^*(m) = \mathbf{0} \tag{2.3}$$

for every function $g : N \to N$ *and* $s \in \{0, 1, \ldots, d-1\}$, *where* $d = d_1 d_2$ *and*

$$f(s-m) = \begin{cases} s - m & \text{if } s \geqslant m, \\ d + s - m & \text{if } s < m. \end{cases}$$

Proof. Let

$$\mathbf{B}_s = (b_{s;ij}) \quad \text{with } i = 1, 2, \ldots, k_1, j = 1, 2, \ldots, k_2,$$

$$\mathbf{Q}^*(n) = (q^*_{ij}(n)) \quad \text{with } i, j = 1, 2, \ldots, k_1,$$

$$\mathbf{X}(n) = (x_{ij}(n)) \quad \text{with } i = 1, 2, \ldots, k_1, \ j = 1, 2, \ldots, k_2,$$

$$\mathbf{Y}^*(n) = (y^*_{ij}(n)) \quad \text{with } i, j = 1, 2, \ldots, k_2,$$

$$\mathbf{c}_j = (c_{i,j}) \quad \text{with } i = 1, 2, \ldots, k_1 \qquad (j = 0, 1, \ldots, d_2 - 1).$$

Also let

$$\mathbf{Q}^*_{k,h}(n) = (q^*_{ij}(n)) \qquad \text{for } i \in_{\mathbf{Q}} C_k,\ j \in_{\mathbf{Q}} C_h \text{ with } k, h \in \{0, 1, \ldots, d_1 - 1\},$$

$$\mathbf{Y}^*_{k,h}(n) = (y^*_{ij}(n)) \qquad \text{for } \ i \in_{\mathbf{Y}} C_k,\ j \in_{\mathbf{Y}} C_h \text{ with } k, h \in \{0, 1, \ldots, d_2 - 1\}.$$

Now, it is known (Iosifescu, 1979) that $\mathbf{Q}^*_{k,h}(n) \neq \mathbf{0}$, $k, h \in \{0, 1, \ldots, d_1\}$, only if $k + n = h \pmod{d_1}$, or equivalently, if the remainder of the division $(k + n)/d_1$ is equal to h [symbolize $h = r(k + n|d_1)$]. It also known that the matrix $\mathbf{Q}^*_{k,h}(n)$ is stable for $h = r(k + n|d_1)$; thus every column of it has identical entries. Hence $\mathbf{Q}^*_{k,h}(n)$ can be expressed in the form

$$\mathbf{Q}^*_{k,h}(n) = (q^*_{\cdot m}\mathbf{1}) \qquad \text{with} \quad m \in {}_{\mathbf{Q}}C_{r(k+n|d_1)}, \tag{2.4}$$

where the constants $q^*_{\cdot m} \in R$.

Similarly we get that $\mathbf{Y}^*_{k,h}(n) \neq \mathbf{0}$ only if $h = r(k + n|d_2)$. Then [for $h = r(k + n|d_2)$]

$$\mathbf{Y}^*_{k,h}(n) = (y^*_{\cdot m}\mathbf{1}) \qquad \text{with} \quad m \in {}_{\mathbf{Y}}C_{r(k+n|d_2)}, \tag{2.5}$$

where $y^*_{\cdot m} \in R$.

Now, let $r_1 = r(f(s-m) + i|d_1)$ and $r_2 = r(j - m|d_2)\,(s, m, i, j \in N)$. From (2.3), (2.4), and (2.5) we get

$$\begin{aligned} b_{s;i,j} &= \sum_{m=0}^{d-1} \sum_{k=1}^{k_1} q^*_{ik}(f(s-m)) \sum_{h=1}^{k_2} x_{kh}(g(m))\, y^*_{hj}(m) \\ &= \sum_{m=0}^{d-1} \sum_{k \in {}_{\mathbf{Q}}C_{r_1}} q^*_{ik}(f(s-m)) \sum_{h \in {}_{\mathbf{Y}}C_{r_2}} x_{kh}(g(m))\, y^*_{hj}(m) \\ &= \sum_{m=0}^{d-1} \sum_{k \in {}_{\mathbf{Q}}C_{r_1}} q^*_{\cdot k} \sum_{h \in {}_{\mathbf{Y}}C_{r_2}} x_{kh}(g(m))\, y^*_{\cdot j}. \end{aligned}$$

Let $m = d_1 u + v$, where $u \in \{0, 1, \ldots, d_2 - 1\}$, $v \in \{0, 1, \ldots, d_1 - 1\}$, and denote $r_3 = r(s - ud_1 - v + i|d_1)$ $[= r(s - v + i|d_1]$ and $r_4 = r(j - ud_1 - v|d_2)$ for every $s, u, v, i, j \in N$. Then, on account of (2.2), the last relation becomes

$$\begin{aligned} b_{s;i,j} &= y^*_{\cdot j} \sum_{v=0}^{d_1-1} \sum_{u=0}^{d_2-1} \sum_{k \in {}_{\mathbf{Q}}C_{r_3}} q^*_{\cdot k} \sum_{h \in {}_{\mathbf{Y}}C_{r_4}} x_{kh}(g(d_1 u + v)) \\ &= y^*_{\cdot j} \sum_{v=0}^{d_1-1} \sum_{k \in {}_{\mathbf{Q}}C_{r_3}} q^*_{\cdot k} \sum_{u=0}^{d_2-1} c_{k,r_4}. \end{aligned} \tag{2.6}$$

Now, since the integers $0, 1, \ldots, d_2 - 1$ form a complete system of distinct representatives of the congruence classes modulo d_2 and $(d_1, d_2) = 1$, then the integers $(j - v) - 0d_1, (j - v) - 1d_1, \ldots, (j - v) - (d_2 - 1)d_1$ also form a complete system of distinct representatives of the congruence classes modulo d_2 (Stark, 1970, Theorem 3.12). Hence

$$\sum_{u=0}^{d_2-1} c_{k,r_4}\left(=\sum_{u=0}^{d_2-1} c_{k,r_4(u)}\right)=\sum_{j=1}^{k_2} x_{kj}(\cdot)=0,$$

since $\mathbf{X}(t)\mathbf{1}=\mathbf{0}$ for every $t \in N$. Thus (2.6) implies $b_{s;i,j}=0$, for every $i \in \{1,2,\ldots,k_1\}$, $j \in \{1,2,\ldots,k_2\}$, and $s \in \{0,1,\ldots,d-1\}$. ■

By multiplying both sides of (2.3) by $\mathbf{1}$ we easily arrive at the following result:

COROLLARY 2.1. *If the conditions of Lemma 2.2 hold, then*

$$\sum_{m=0}^{d-1} \mathbf{Q}^*(f(s-m))\mathbf{X}(g(m))\mathbf{1}=\mathbf{0}.$$

In what remains we use as the norm $\|A\|$ of a matrix $\mathbf{A}=(a_{ij})_{i,j\in G}$ the quantity $\sup_{i\in G}\sum_{j\in G}a_{ij}$, and we define the *incidence matrix* of $\mathbf{A}$ as the matrix $\tilde{\mathbf{A}}=(\tilde{a}_{ij})_{i,j\in G}$ for which $\tilde{a}_{ij}=1$ if $a_{ij}>0$ and $\tilde{a}_{ij}=0$ if $a_{ij}=0$.

Moreover, let $\{\mathbf{A}(t): t=0,1,\ldots\}$ be a sequence of matrices, and $\mathbf{A}$ a matrix such that $\lim_{t\to\infty}\|\mathbf{A}(t)-\mathbf{A}\|=0$. We say that the rate of convergence of the sequence $\{\mathbf{A}(t): t=0,1,\ldots\}$ to the matrix $\mathbf{A}$ is *geometric* if there are constants c and b, $0<b<1$, such that $\|\mathbf{A}(t)-\mathbf{A}\|<cb^t$ for every $t\in N$.

In the proofs that follow we will make use of the following lemma:

LEMMA 2.4 (Iosifescu, 1979). *If $\mathbf{A}$ is a square matrix such that $\lim_{t\to\infty}\mathbf{A}^t=\mathbf{0}$, then the matrix $(\mathbf{I}-\mathbf{A})^{-1}$ exists and $(\mathbf{I}-\mathbf{A})^{-1}=\sum_{t=0}^{\infty}\mathbf{A}^t$.*

We now provide in the (suitable) form of a theorem the following proposition, which is due to Bowerman (1974) and to Bowerman, David, and Isaacson (1977).

THEOREM 2.1. *Let $\{\mathbf{Q}(t): t=0,1,\ldots\}$ be a sequence of finite, irreducible, stochastic matrices such that $\lim_{t\to\infty}\|\mathbf{Q}(t)-\mathbf{Q}\|=0$, where $\mathbf{Q}$ is a stochastic matrix of period d, the rate of convergence is geometric, and $\tilde{\mathbf{Q}}(t)=\tilde{\mathbf{Q}}$ for $t\geqslant T<\infty$. If $s, t_n\in N$ and $s=n_0d+v$, $t_n=nd+r$, $0\leqslant v<d$, $0\leqslant r<d$, then the sequences $Q(s,t_n)$ converge as $n\to\infty$ and the rate of convergence is geometric, for every $s\in N$.*

A corollary of the above theorem, useful in the proof of the basic Theorem 2.2, is the following:

COROLLARY 2.2. *Let* $\{\mathbf{Q}(t): t = 0, 1, \ldots\}$ *be a sequence of finite, irreducible, stochastic matrices such that* $\lim_{t\to\infty}\|\mathbf{Q}(t) - \mathbf{Q}\| = 0$, *where* $\mathbf{Q}$ *is a stochastic matrix of period* d, *and* $\tilde{\mathbf{Q}}(t) = \tilde{\mathbf{Q}}$ *for every* $t \in N$. *If* $s, t_n \in N$ *and* $s = n_0 d + v$, $t_n = nd + r$, $0 \leqslant v < d$, $0 \leqslant r < d$, *then* $\lim_{t\to\infty}\mathbf{Q}(s, t_n) = \mathbf{Q}^*\mathbf{Q}^{f(r-v+1)}$, *where* $\mathbf{Q}^* = \lim_{n\to\infty}\mathbf{Q}^{nd}$.

The following basic theorem is now provided.

THEOREM 2.2. *Let* $\mathbf{V}(t)$ *be a sequence of* $\mathbf{V}$*-matrices (of the form* (1.1)*), where the matrices* $\mathbf{Q}(t)$ *and* $\mathbf{Y}(t)$ *are irreducible and periodic with periods* d_1 *and* d_2 *respectively, and* $(d_1, d_2) = 1$. *If*

(1) *the relation* (2.2) *holds,*

(2) *there are matrices* $\mathbf{Q}, \mathbf{Y}$, *and* $\mathbf{X}$ *such that* $\lim_{t\to\infty}\mathbf{Q}(t) = \mathbf{Q}$, $\lim_{t\to\infty}\mathbf{Y}(t) = \mathbf{Y}$, $\lim_{t\to\infty}\mathbf{X}(t) = \mathbf{X}$, *with* $\tilde{\mathbf{Q}}(t) = \tilde{\mathbf{Q}}$, $\tilde{\mathbf{Y}}(t) = \tilde{\mathbf{Y}}$ *for every* $t \in N$, *and*

(3) *the rate of convergence of the sequences* $\mathbf{Q}(t)$ *and* $\mathbf{Y}(t)$ *is geometric,*

then the sequence $\mathbf{V}(t, t+n)$ *splits into* $d_1 d_2$ *subsequences which converge geometrically fast as* $n \to \infty$ *(the limits are calculated in Section* 3*).*

Proof. From Tsaklidis and Vassiliou (1990) we get that

$$\mathbf{V}(t, t+n) = \left[\begin{array}{c|c} \mathbf{Q}(t, t+n) & \mathbf{S}(t, t+n) \\ \hline \mathbf{0} & \mathbf{Y}(t, t+n) \end{array}\right],$$

with

$$\mathbf{S}(t, t+n) = \sum_{j=0}^{n} \mathbf{Q}(t, t+n-j-1)\mathbf{X}(t+n-j)\mathbf{Y}(t+n+1-j, t+n) \tag{2.7}$$

[with $\mathbf{Q}(t, t-1) = \mathbf{I}$, $\mathbf{Y}(t, t-1) = \mathbf{I}$ for every $t \in N$]. The sequence $\mathbf{Q}(t, t+n)$ splits (Theorem 2.1, Corollary 2.2) into d_1 convergent subsequences of the form $\mathbf{Q}(t, nd_1 + r)$, $r = 0, 1, \ldots, d_1 - 1$, and the sequence $\mathbf{Y}(t, t+n)$ splits into d_2 convergent subsequences of the form $\mathbf{Y}(t, nd_2 + v)$, $v = 0, 1, \ldots, d_2 - 1$. Thus, for the sequence $\mathbf{V}(t, t+n)$ the blocks containing the sequences $\mathbf{Q}(t, t+n)$, $\mathbf{Y}(t, t+n)$ can be thought as a sequence which splits into k convergent subsequences, where k is the least common multiple of d_1 and d_2. In order to prove that the sequence $\mathbf{V}(t, t+n)$ splits into $d = d_1 d_2$ subsequences with $(d_1, d_2) = 1$ it is sufficient to prove that the remaining block sequence containing $\mathbf{S}(t, t+n)$ splits into $d = d_1 d_2$ converging subsequences when $(d_1, d_2) = 1$.

Let

$$\mathbf{Q}^*(m) = \lim_{k\to\infty} \mathbf{Q}(t, t + kd_1 + m - 1),$$

$$\mathbf{Y}^*(n) = \lim_{k\to\infty} \mathbf{Y}(t, t + kd_2 + n - 1),$$

$${}_{\mathbf{Q}}\mathbf{D}(t, t + kd_1 + m) = \mathbf{Q}(t, t + kd_1 + m) - \mathbf{Q}^*(m + 1),$$

$${}_{\mathbf{Y}}\mathbf{D}(t, t + kd_1 + n) = \mathbf{Y}(t, t + kd_1 + n) - \mathbf{Y}^*(n + 1),$$

where $m \in \{0, 1, \ldots, d_1 - 1\}$, $n \in \{0, 1, \ldots, d_2 - 1\}$. It is apparent that the definitions of $\mathbf{Q}^*(m), \mathbf{Y}^*(n), {}_{\mathbf{Q}}\mathbf{D}(t, t + kd_1 + m), {}_{\mathbf{Y}}\mathbf{D}(t, t + kd_1 + n)$ can be extended to every $m, n \in N$. Then, if $m = pd_1 + r_1$, $p \in N$, $0 \leqslant r_1 < d_1$ and if $n = qd_2 + r_2, q \in N, 0 \leqslant r_2 < d_2$, we easily get $\mathbf{Q}^*(m) = \mathbf{Q}^*(r_1)$ and $\mathbf{Y}^*(n) = \mathbf{Y}^*(r_2)$, $m, n \in N$.

Now for the convergent subsequences of $\mathbf{Q}(t, t + n)$ and $\mathbf{Y}(t, t + n)$ there are (Theorem 2.1) constants c_1, c_2, b_1, b_2 with $0 < b_1, b_2 < 1$ such that

$$\|{}_{\mathbf{Q}}\mathbf{D}(t, t + kd_1 + m)\| \leqslant c_1 b_1^{kd_1+m+1} \quad \text{and}$$

$$\|{}_{\mathbf{Y}}\mathbf{D}(t, t + kd_2 + n)\| \leqslant c_2 b_2^{kd_2+n+1} \tag{2.8}$$

for every $t \in N$. From (2.7) we get

$$\begin{aligned}\mathbf{S}(t, nd + s) = \sum_{m=0}^{d-1} \Bigg\{ \sum_{j=0}^{n-1} & \mathbf{Q}(t, t + (n - j)d + s - m - 1) \\ & \times \mathbf{X}(t + (n - j)d + s - m) \\ & \times \mathbf{Y}(t + (n - j)d + s - m + 1, t + nd + s) \Bigg\} \\ + \sum_{m=0}^{s} & \mathbf{Q}(t, t + s - m - 1)\mathbf{X}(t + s - m) \\ & \times \mathbf{Y}(t + s - m + 1, t + nd + s)\end{aligned} \tag{2.9}$$

$$= \sum_{m=0}^{d-1} \Bigg\{ \sum_{j=0}^{n-1} {}_{\mathbf{Q}}\mathbf{D}(t, t + (n-j)d + s - m - 1)$$

$$\times \mathbf{X}(t + (n-j)d + s - m)$$

$$\times {}_{\mathbf{Y}}\mathbf{D}(t + (n-j)d + s - m + 1, t + nd + s)$$

$$+ \sum_{j=0}^{n-1} {}_{\mathbf{Q}}\mathbf{D}(t, t + (n-j)d + s - m - 1)$$

$$\times \mathbf{X}(t + (n-j)d + s - m)\mathbf{Y}^*(m)$$

$$+ \sum_{j=0}^{n-1} \mathbf{Q}^*(f(s-m))\mathbf{X}(t + (n-j)d + s - m)$$

$$\times {}_{\mathbf{Y}}\mathbf{D}(t + (n-j)d + s - m + 1, t + nd + s)$$

$$+ \sum_{j=0}^{n-1} \mathbf{Q}^*(f(s-m))\mathbf{X}(t + (n-j)d + s - m)\mathbf{Y}^*(m) \Bigg\}$$

$$+ \sum_{m=0}^{s} \mathbf{Q}(t, t + s - m - 1)\mathbf{X}(t + s - m)$$

$$\times \mathbf{Y}(t + s - m + 1, t + nd + s). \tag{2.10}$$

Now, by (2.8) and since the sequence $\mathbf{X}(t)$ is bounded [$\mathbf{X}(t)$ is convergent], it can be proved by following the steps of the proof of Theorem 2.2 in Tsaklidis and Vassiliou (1990) that the first sum in (2.10) tends to $\mathbf{0}$, as $n \to \infty$, geometrically fast and uniformly in t. Furthermore, according to Lemma 2.2 [note that $(d_1, d_2) = 1$ by hypothesis], the fourth sum in (2.10) is the null matrix. As regards the second sum in (2.10), the sequence $\mathbf{X}(t)$ is bounded; thus there exists $c \in R^+$ such that $\|\mathbf{X}(t)\| < c$ for every $t \in n$. Then, by (2.8),

$$\|{}_{\mathbf{Q}}\mathbf{D}(t, t + (n-j)d + s - m - 1)\mathbf{X}(t + (n-j)d + s - m)\|$$

$$\leqslant \|{}_{\mathbf{Q}}\mathbf{D}(t, t + (n-j)d + s - m - 1)\| \|\mathbf{X}(t + (n-j)d + s - m)\|$$

$$\leqslant (c_1 c) b_1^{(n-j)d + s - m} \qquad [(n-j)d + s - m \geqslant 0],$$

which implies that the sequence $\sum_{j=0}^{n-1} {}_{\mathbf{Q}}\mathbf{D}(t, t + (n - j)d + s - m - 1)\mathbf{X}(t + (n - j)d + s - m)$ converges—geometrically fast and uniformly in t—as $n \to \infty$, for every $s, m \in N$. Similarly, it can be proved that the third sum in (2.10) converges geometrically fast and uniformly in t as $n \to \infty$.

Thus, from (2.10) we get that the sequence $\mathbf{S}(t, nd + s)$ converges as $n \to \infty$ for every $t \in N$, but not uniformly in t (since the last sum depends on t). ∎

REMARK 2.3. It can be shown by counterexamples that if $(d_1, d_2) \neq 1$, or if $(d_1, d_2) = 1$ but (2.2) fails, then—generally—the conclusion of Theorem 2.2 also fails.

3. CALCULATION OF $\lim_{n\to\infty} \mathbf{S}(t, nd + s)$

In the previous theorem we proved that the sequence $\mathbf{V}(t, t + n)$ splits into $d = d_1 d_2$ converging subsequences. In order to find the limits of the subsequences, what remains is to provide a method to calculate the $\lim_{n\to\infty} \mathbf{S}(t, nd + s)$ $(s \in \{0, 1, \ldots, d - 1\})$.

Using (2.7) we get

$$
\begin{aligned}
&\mathbf{S}(t, (n + 1)d + s)\\
&\quad = \sum_{m=0}^{d-1} \sum_{j=0}^{n} \mathbf{Q}(t, t + (n + 1 - j)d + s - m - 1)\\
&\qquad \times \mathbf{X}(t + (n + 1 - j)d + s - m)\\
&\qquad \times \mathbf{Y}(t + (n + 1 - j)d + s - m + 1, t + (n + 1)d + s)\\
&\qquad + \sum_{m=0}^{s} \mathbf{Q}(t, t + s - m - 1)\mathbf{X}(t + s - m)\\
&\qquad\quad \times \mathbf{Y}(t + s - m + 1, t + (n + 1)d + s)\\
&\quad = \sum_{m=0}^{d-1} \sum_{j=1}^{n} \mathbf{Q}(t, t + (n + 1 - j)d + s - m - 1)\\
&\qquad \times \mathbf{X}(t + (n + 1 - j)d + s - m)
\end{aligned}
$$

$$\times \mathbf{Y}(t+(n+1-j)d+s-m+1, t+(n+1)d+s)$$

$$+\sum_{m=0}^{d-1} \mathbf{Q}(t, t+(n+1)d+s-m-1)$$

$$\times \mathbf{X}(t+(n+1)d+s-m)$$

$$\times \mathbf{Y}(t+(n+1)d+s-m+1, t+(n+1)d+s)$$

$$+\sum_{m=0}^{s} \mathbf{Q}(t, t+s-m-1)\mathbf{X}(t+s-m)$$

$$\times \mathbf{Y}(t+s-m+1, t+(n+1)d+s). \tag{3.1}$$

Now, by (2.9), we have

$$\sum_{m=0}^{d-1}\sum_{j=1}^{n} \mathbf{Q}(t, t+(n+1-j)d+s-m-1)\mathbf{X}(t+(n+1-j)d+s-m)$$

$$\times \mathbf{Y}(t+(n+1-j)d+s-m+1, t+(n+1)d+s)$$

$$= \left\{\sum_{m=0}^{d-1}\sum_{j=0}^{n-1} \mathbf{Q}(t, t+(n-j)d+s-m-1)\right.$$

$$\left.\times \mathbf{X}(t+(n-j)d+s-m)\mathbf{Y}(t+(n-j)d+s-m+1, t+nd+s)\right\}$$

$$\times \mathbf{Y}(t+nd+s+1, t+(n+1)d+s)$$

$$= \left\{\mathbf{S}(t, nd+s) - \sum_{m=0}^{s} \mathbf{Q}(t, t+s-m-1)\right.$$

$$\left.\times \mathbf{X}(t+s-m)\mathbf{Y}(t+s-m+1, t+nd+s)\right\}$$

$$\times \mathbf{Y}(t+nd+s+1, t+(n+1)d+s).$$

Then (3.1) becomes

$$\mathbf{S}(t,(n+1)d+s) = \mathbf{S}(t, nd+s)\mathbf{Y}(t+nd+s+1, t+(n+1)d+s)$$

$$+\sum_{m=0}^{d-1} \mathbf{Q}(t, t+(n+1)d+s-m-1)$$

$$\times \mathbf{X}(t + (n + 1)d + s - m)$$
$$\times \mathbf{Y}(t + (n + 1)d + s - m$$
$$+ 1, t + (n + 1)d + s). \tag{3.2}$$

In (3.2) let $n \to \infty$ and denote $\mathbf{S}^*(t, s) = \lim_{n\to\infty}\mathbf{S}(t, nd + s)$ to get

$$\mathbf{S}^*(t, s) = \mathbf{S}^*(t, s)\mathbf{Y}^d + \sum_{m=0}^{d-1} \mathbf{Q}^*(f(s - m))\mathbf{X}\mathbf{Y}^m,$$

or

$$\mathbf{S}^*(t, s)(\mathbf{I} - \mathbf{Y}^d) = \mathbf{C}(s), \tag{3.3}$$

where $\mathbf{C}(s) = \sum_{m=0}^{d-1}\mathbf{Q}^*(f(s - m))\mathbf{X}\mathbf{Y}^m$.

In order to evaluate $\mathbf{S}^*(t, s)$ from (3.3), since $\mathbf{Y}$ is a periodic stochastic matrix of period d_2 and d is a multiple of d_2, the matrix $\underline{\mathbf{Y}}^d$ can be written in the form

$$\begin{array}{c} \\ {}_{\mathbf{Y}}C_0 \\ {}_{\mathbf{Y}}C_1 \\ \vdots \\ {}_{\mathbf{Y}}C_{d_2-1} \end{array} \begin{array}{c} \begin{array}{cccc} {}_{\mathbf{Y}}C_0 & {}_{\mathbf{Y}}C_1 & \cdots & {}_{\mathbf{Y}}C_{d_2-1} \end{array} \\ \begin{bmatrix} \mathbf{Y}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Y}_{d_2-1} \end{bmatrix} \end{array}; \tag{3.4}$$

thus

$$\mathbf{I} - \mathbf{Y}^d = \begin{bmatrix} \mathbf{I} - \mathbf{Y}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} - \mathbf{Y}_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} - \mathbf{Y}_{d_2-1} \end{bmatrix}. \tag{3.5}$$

Let $\mathbf{S}^*(t, s) = (\mathbf{S}^*_{u,v}(t, s))$, with $u = 0, 1, \dots, d_1 - 1$, $v = 0, 1, \dots, d_2 - 1$, where $\mathbf{S}^*_{u,v}(t, s)$ are $n({}_{\mathbf{Q}}C_u) \times n({}_{\mathbf{Y}}C_v)$ matrices. Similarly let $\mathbf{C}(s) = \mathbf{C}_{u,v}(s))$, with $u = 0, 1, \dots, d_1 - 1$, $v = 0, 1, \dots, d_2 - 1$ $(s \in \{0, 1, \dots, d - 1\})$, where $\mathbf{C}_{u,v}(s)$ are $n({}_{\mathbf{Q}}C_u) \times n({}_{\mathbf{Y}}C_v)$ matrices. Then, by (3.5), Equation (3.3) becomes

$$\mathbf{S}^*_{u,v}(t, s) \cdot (\mathbf{I} - \mathbf{Y}_v) = \mathbf{C}_{u,v}(s),$$

or

$$\mathbf{S}^*_{u,v}(t,s)\mathbf{A}_v = \mathbf{C}_{u,v}(s), \qquad u = 0,1,\dots,d_1 - 1, \quad v = 0,1,\dots,d_2 - 1, \tag{3.6}$$

where $\mathbf{A}_v = \mathbf{I} - \mathbf{Y}_v$.

In order to evaluate the matrices $\mathbf{S}^*_{u,v}(t,s)$—and thus $\mathbf{S}^*(t,s)$—from (3.6), Lemma 3.1 is provided. In what follows consider $\mathbf{X}(t)$ partitioned [as in (2.1)] in the form $\mathbf{X}(t) = (\mathbf{X}_{i,j}(t))$, $i = 0,1,\dots,d_1 - 1$, $j = 0,1,\dots,d_2 - 1$, where $\mathbf{X}_{i,j}(t)$ are $n(_{\mathbf{Q}}C_i) \times n(_{\mathbf{Y}}C_j)$ submatrices. Then, by Theorem 2.2 [Equation (2.2)], there are column vectors $\mathbf{c}_{i,j}$ such that

$$\mathbf{X}_{i,j}(t)\mathbf{1} = \mathbf{c}_{i,j} \qquad \text{for every} \quad t \in N, \quad i = 0,1,\dots,d_1 - 1,$$
$$j = 0,1,\dots,d_2 - 1. \tag{3.7}$$

LEMMA 3.1.

$$\mathbf{S}^*_{u,v}(t,s)\mathbf{1} = \sum_{m=0}^{d-1}\sum_{j=0}^{\infty} {}_{\mathbf{Q}}\mathbf{D}_{u,r_1}(t, t + jd + s - m - 1)\mathbf{c}_{r_1,r_2}$$
$$+ \sum_{m=0}^{s} \mathbf{Q}_{u,r_1}(t, t + s - m - 1)\mathbf{c}_{r_1,r_2} \tag{3.8}$$

for every $u \in \{0,1,\dots,d_1 - 1\}$, $v \in \{0,1,\dots,d_2 - 1\}$, where $r_1 = r(s - m + u|d_1)$ and $r_2 = r(v - m|d_2)$.

Proof. As is seen by the examination of the sums of (2.10), the first (double) sum tends to $\mathbf{0}$ as $n \to \infty$, and the fourth sum is also the zero matrix. Thus multiply (2.10) by $\mathbf{1}$ and let $n \to \infty$ to get

$$\mathbf{S}^*_{u,v}(t,s)\cdot\mathbf{1}$$
$$= \sum_{m=0}^{d-1}\Bigg\{\lim_{n\to\infty}\sum_{j=0}^{n-1} {}_{\mathbf{Q}}\mathbf{D}_{u,r_1}(t, t + (n-j)d + s - m - 1)$$
$$\times\mathbf{X}_{r_1,r_2}(t + (n-j)d + s - m)\mathbf{Y}^*_{r_2,v}(m)\Bigg\}\mathbf{1}$$

$$+\sum_{m=0}^{d-1} \mathbf{Q}^*_{u,r_1}(f(s-m))$$

$$\times\left\{\lim_{n\to\infty}\sum_{j=0}^{n-1}\mathbf{X}_{r_1,r_2}(t+(n-j)d+s-m)\right.$$

$$\left.\times\,{}_{\mathbf{Y}}\mathbf{D}_{r_2,v}(t+(n-j)d+s-m+1,t+nd+s)\right\}\mathbf{1}$$

$$+\sum_{m=0}^{s}\mathbf{Q}_{u,r_1}(t,t+s-m-1)\mathbf{X}_{r_1,r_2}(t+s-m)$$

$$\times\left\{\lim_{n\to\infty}\mathbf{Y}_{r_2,v}(t+s-m+1,t+nd+s)\mathbf{1}\right\}. \tag{3.9}$$

By definition of ${}_{\mathbf{Y}}\mathbf{D}(t,n)$ it is obvious that ${}_{\mathbf{Y}}\mathbf{D}(t,n)\mathbf{1}=\mathbf{0}$ for every $n \geqslant t$; thus the second sum in (3.9) vanishes. Then, by (3.7), Equation (3.9) leads to (3.8). ∎

COROLLARY 3.1. *For large t (denote $t \gg$) one has*

$$\mathbf{S}^*_{u,v}(t,s)\cdot\mathbf{1}=\sum_{m=0}^{d-1}(\mathbf{Q}^{d+s-m})_{u,r_1}\cdot\left\{\left[I-\left[\mathbf{Q}^d-\mathbf{Q}^*(0)\right]\right\}^{-1}\right\}_{r_1,r_2}\cdot\mathbf{c}_{r_1,r_2}$$

$$+\sum_{m=0}^{s}(\mathbf{Q}^{s-m})_{u,r_1}\mathbf{c}_{r_1,r_2}, \tag{3.10}$$

where $u\in\{0,1,\ldots,d_1-1\}$, $v\in\{0,1,\ldots,d_2-1\}$, $r_1=r(s-m+u|d_1)$, and $r_2=r(v-m|d_2)$.

Proof. From Lemma 2.1 we easily get

$$\mathbf{Q}^{d+s-m}\mathbf{Q}^*(0)=\mathbf{Q}^*(f(s-m)); \tag{3.11}$$

then for $t \gg$,

$$\begin{aligned}{}_{\mathbf{Q}}\mathbf{D}(t,t+jd+s-m-1)&=\mathbf{Q}(t,t+jd+s-m-1)-\mathbf{Q}^*(f(s-m))\\&=\mathbf{Q}^{jd+s-m}-\mathbf{Q}^*(f(s-m))\\&=\mathbf{Q}^{d+s-m}\{\mathbf{Q}^{(j-1)d}-\mathbf{Q}^*(0)\}\qquad(j\geqslant 1).\end{aligned}$$

Hence

$$
{}_{Q}\mathbf{D}_{u,r_1}(t, t + jd + s - m - 1) = (\mathbf{Q}^{d+s-m})_{u,r_1}\{\mathbf{Q}^{(j-1)d} - \mathbf{Q}^*(0)\}_{r_1,r_1} \tag{3.12}
$$

for every $j \geqslant 1$, $u \in \{0, 1, \ldots, d_1 - 1\}$, where $r_1 = r(s - m + u|d_1)$.

Denote by $\mathbf{b}$ the double sum in (3.8). Then, by (3.12), for $t \succ$ we get

$$
\mathbf{b} = \sum_{m=0}^{d-1} (\mathbf{Q}^{d+s-m})_{u,r_1} \cdot \left\{ \sum_{j=0}^{\infty} \{\mathbf{Q}^{jd} - \mathbf{Q}^*(0)\} \right\}_{r_1,r_1} \cdot \mathbf{c}_{r_1,r_2}. \tag{3.13}
$$

Since $\mathbf{Q}^*(0) = \lim_{j\to\infty} \mathbf{Q}^{jd}$ (by definition), then—by Lemma 2.4—(3.13) implies

$$
\mathbf{b} = \sum_{m=0}^{d-1} (\mathbf{Q}^{d+s-m})_{u,r_1} \cdot \left\{ \left\{ \mathbf{I} - \left[\mathbf{Q}^d - \mathbf{Q}^*(0)\right]\right\}^{-1} - \mathbf{Q}^*(0) \right\}_{r_1,r_2} \cdot \mathbf{c}_{r_1,r_2}, \tag{3.14}
$$

and on account of (3.11) and Corollary 2.1, (3.14) becomes

$$
\mathbf{b} = \sum_{m=0}^{d-1} (\mathbf{Q}^{d+s-m})_{u,r_1} \cdot \left\{ \left\{ \mathbf{I} - \left[\mathbf{Q}^d - \mathbf{Q}^*(0)\right]\right\}^{-1} \right\}_{r_1,r_2} \cdot \mathbf{c}_{r_1,r_2}.
$$

From the last relation and (3.8) for $t \succ$, (3.10) follows. ■

Now, since the matrices $\mathbf{Y}_i$, $i = 0, 1, \ldots, d_2 - 1$, in (3.4) are regular (Iosifescu, 1979, §4.4.2), Equation (3.6) can be solved in terms of $\mathbf{S}^*(t, s)$ by following the steps of solving Equation (3.2) in Tsaklidis and Vassiliou (1990). Hence

$$
\mathbf{S}^*_{u,v}(t, s) = \mathbf{C}_{u,v}(s)\mathbf{K}_v + \mathbf{Z}(\mathbf{I} - \mathbf{H}_{\mathbf{A}_v}), \qquad u = 0, 1, \ldots, d_1 - 1,
$$

$$
v = 0, 1, \ldots, d_2 - 1, \tag{3.15}
$$

where $\mathbf{H}_{\mathbf{A}_v}$ is the Hermite canonical form of the matrix $\mathbf{A}_v$, $\mathbf{K}_v$ is a generalized inverse of $\mathbf{A}_v$ such that $\mathbf{A}_v\mathbf{K}_v = \mathbf{H}_{\mathbf{A}_v}$ ($v = 0, 1, \ldots, d_2 - 1$), and

$\mathbf{Z}$ is any $n({}_{\mathbf{Q}}C_u) \times n({}_{\mathbf{Y}}C_v)$ real matrix. The matrix $\mathbf{H}_{\mathbf{A}_v}$ is given by

$$\mathbf{H}_{\mathbf{A}_v} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ h_{c_v,1} & h_{c_v,2} & \cdots & h_{c_v,c_v-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{3.16}$$

where the index c_v $[1 < c_v \leqslant n({}_{\mathbf{Y}}C_v)]$ is unique for every matrix $\mathbf{H}_{\mathbf{A}_v}$, and where $h_{c_v,1}, h_{c_v,2}, \ldots, h_{c_v,c_v-1} \in R$ with $\sum_{i=1}^{c_v-1} h^2_{c_v,i} \neq 0$ for every $v \in \{0, 1, \ldots, d_2 - 1\}$. Denote by $\mathbf{a}'_v$ the nonnull row vector of $\mathbf{I} - \mathbf{H}_{\mathbf{A}_v}$. Then (3.15) becomes

$$\mathbf{S}^*_{u,v}(t,s) = \mathbf{C}_{u,v}(s)\mathbf{K}_v + \mathbf{z}\mathbf{a}'_v, \qquad u = 0, 1, \ldots, d_1 - 1,$$
$$v = 0, 1, \ldots, d_2 - 1, \tag{3.17}$$

where $\mathbf{z}$ is any $n({}_{\mathbf{Q}}C_u) \times 1$ vector, $s \in \{0, 1, \ldots, d - 1\}$.

Now, the solution of Equation (3.17) in terms of $\mathbf{S}^*_{u,v}(t, s)$ must be—by Theorem 2.2—unique. Thus, the problem of evaluating $\mathbf{S}^*_{u,v}(t, s)$ reduces to the evaluation of the appropriate vector $\mathbf{z}$ in (3.17). In view of (3.16), multiply (3.17) by $\mathbf{1}$ to get

$$\mathbf{S}^*_{u,v}(t,s)\mathbf{1} = \mathbf{C}_{u,v}(s)\mathbf{K}_v\mathbf{1} + \mathbf{z}\left(1 - \sum_{j=1}^{c_v-1} h_{c_v,j}\right). \tag{3.18}$$

Note that $\mathbf{C}_{u,v}(s) = \sum_{m=0}^{d-1} \mathbf{Q}^*_{u,r_1}(f(s-m))\mathbf{X}_{r_1,r_2}\mathbf{Y}_{r_2,v}$, $r_1 = r(s - m + u|d_1)$, $r_2 = r(v - m|d_2)$, is a stable matrix, since $\mathbf{Q}^*_{u,r_1}(f(s - m))$ is stable (Iosifescu, 1979). Thus $\mathbf{C}_{u,v}(s)\mathbf{K}_v$ is also a stable matrix, and it can be written as

$$\mathbf{C}_{u,v}(s)\mathbf{K}_v = \mathbf{1}\mathbf{p}'(s; u, v), \tag{3.19}$$

where $\mathbf{1}$ is the $n({}_{\mathbf{Q}}C_u) \times 1$ column vector of 1's and $\mathbf{p}'(s; u, v)$ is a $1 \times n({}_{\mathbf{Y}}C_v)$ row vector. Moreover, $\sum_{j=1}^{c_v-1} h_{c_v,j} \neq 1$ (Tsaklidis and Vassiliou, 1990). Hence, by (3.19), Equation (3.18) implies

$$\mathbf{z} = \left(1 - \sum_{j=1}^{c_v - 1} h_{c_v, j}\right)^{-1} \{\mathbf{S}^*_{u,v}(t,s)\mathbf{1} - p(s;u,v)\mathbf{1}\}', \tag{3.20}$$

where $p(s;u,v) = \mathbf{p}'(s;u,v)\mathbf{1}$.

From (3.17), (3.19), and (3.20) follows

$$\mathbf{S}^*_{u,v}(t,s) = \mathbf{1p}'(s;u,v) + \left(1 - \sum_{j=1}^{c_v - 1} h_{c_v, j}\right)^{-1} \{\mathbf{S}^*_{u,v}(t,s)\mathbf{1} - p(s;u,v)\mathbf{1}\}\mathbf{a}'_v \tag{3.21}$$

for every $u \in \{0, 1, \dots, d_1 - 1\}$, $v \in \{0, 1, \dots, d_2 - 1\}$, where $\mathbf{S}^*_{u,v}(t,s)\mathbf{1}$ is given by (3.8) [or (3.10) if $t \gg$]. ■

ILLUSTRATION. In what follows we illustrate the results of the present paper by means of an example. To simplify the calculations we'll evaluate $\mathbf{S}^*(t,s)$—using (3.21)—in the case $t \gg$.

Let

$$\mathbf{V}(t) = \left[\begin{array}{c|c} \mathbf{Q}(t) & \mathbf{X}(t) \\ \hline \mathbf{0} & \mathbf{Y}(t) \end{array}\right],$$

where $\mathbf{V}(t)$ are $\mathbf{V}$-matrices such that the conditions of Theorem 2.2 hold, with

$$\mathbf{Q} = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \hline 0.2 & 0.8 & 0 \end{array}\right],$$

$$\mathbf{X} = \left[\begin{array}{ccc|cccc|cc} 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right],$$

and

$$\mathbf{Y} = \left[\begin{array}{ccc|cccc|cc} & \mathbf{0} & & 0.6 & 0.2 & 0.1 & 0.1 & & \mathbf{0} \\ & & & 0.4 & 0.4 & 0.2 & 0 & & \\ & & & 0.1 & 0 & 0.4 & 0.5 & & \\ \hline & & & & & & & 0.6 & 0.4 \\ & \mathbf{0} & & & \mathbf{0} & & & 0.2 & 0.8 \\ & & & & & & & 0.5 & 0.5 \\ & & & & & & & 0.4 & 0.6 \\ \hline 0.6 & 0.2 & 0.2 & & \mathbf{0} & & & & \mathbf{0} \\ 0.4 & 0.2 & 0.4 & & & & & & \end{array}\right].$$

Then $d_1 = 2$, $d_2 = 3$, and $d = 6$. Now,

$$\mathbf{Y}^6 = \left[\begin{array}{ccc|cccc|cc} .4934 & .2 & .3066 & & & & & & \\ .4933 & .2 & .3067 & & \mathbf{0} & & & & \mathbf{0} \\ .4934 & .2 & .3067 & & & & & & \\ \hline & & & .4068 & .1788 & .2119 & .2026 & & \\ & \mathbf{0} & & .4065 & .1787 & .212 & .2028 & & \\ & & & .4067 & .1787 & .212 & .2026 & & \\ & & & .4066 & .1787 & .212 & .2027 & & \\ \hline & \mathbf{0} & & & \mathbf{0} & & & .4668 & .5332 \\ & & & & & & & .4668 & .5332 \end{array}\right].$$

The Hermite canonical forms $\mathbf{H}_{\mathbf{A}_v}$ of the submatrices $\mathbf{A}_v = (\mathbf{I} - \mathbf{Y}^6)_v$ and the generalized inverses $\mathbf{K}_v$, $v = 0, 1, 2$, such that $\mathbf{A}_v\mathbf{K}_v = \mathbf{H}_{\mathbf{A}_v}$ are found to be

$$\mathbf{H}_{\mathbf{A}_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1.6089 & -0.6522 & 0 \end{bmatrix}, \quad \mathbf{K}_0 = \begin{bmatrix} 2.609 & 0.6523 & 1 \\ 1.6088 & 1.6522 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{H}_{\mathbf{A}_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2.0067 & -0.8816 & -1.0461 & 0 \end{bmatrix},$$

$$\mathbf{K}_1 = \begin{bmatrix} 3.007 & 0.8818 & 1.0461 & 1 \\ 2.0064 & 1.8815 & 1.046 & 1 \\ 2.0069 & 0.8817 & 2.0461 & 1 \\ 0 & 0 & -1.0461 & 1 \end{bmatrix},$$

$$\mathbf{H}_{\mathbf{A}_2} = \begin{bmatrix} 1 & 0 \\ -0.8755 & 0 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 1.8755 & 1 \\ 0 & 1 \end{bmatrix};$$

thus

$$\left\{1-\sum_{j=1}^{c_v-1} h_{c_v,j}\right\}^{-1} = \begin{cases} 0.3066 & \text{if} \quad v=0, \\ 0.2027 & \text{if} \quad v=1, \\ 0.5332 & \text{if} \quad v=2, \end{cases}$$

and $\mathbf{a}_0' = (1.6089, 0.6522, 1)$, $\mathbf{a}_1' = (2.0067, 0.8816, 1.0461, 1)$, $\mathbf{a}_2' = (0.8755, 1)$.

Let $s = 5$. In order to evaluate the submatrices $\mathbf{S}^*_{u,v}(t,5)$ from (3.21) notice that

$$\mathbf{C}(5) = \sum_{m=0}^{5} \mathbf{Q}^*(5-m)\mathbf{X}\mathbf{Y}^m$$

$$= \left[\begin{array}{ccc|cccc|cc} -.6027 & .8 & -.1973 & .4257 & .3543 & -.8114 & .0314 & -.1432 & .1432 \\ -.6027 & .8 & -.1973 & .4257 & .3543 & -.8114 & .0314 & -.1432 & .1432 \\ \hline -.322 & .6 & -.27804 & -.4106 & .4854 & .36 & -.4347 & -.547 & .547 \end{array}\right]$$

Now, multiply the submatrices $\mathbf{C}_{u,v}(5)$ by $\mathbf{K}_v$, $v = 0, 1, 2$, to get $\mathbf{1p}'(5; u.v)$. For example, if $u = 0$, $v = 2$, then

$$\mathbf{C}_{0,2}(5) \cdot \mathbf{K}_2 = \begin{bmatrix} -.1432 & .1432 \\ -.1432 & .1432 \end{bmatrix} \begin{bmatrix} 1.8755 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -.2686 & 0 \\ -.2686 & 0 \end{bmatrix} = \underline{\mathbf{1p}}'(5; 0, 2),$$

where $\mathbf{p}'(5; 0, 2) = (-.2686, 0)$.

What remains is to evaluate $\mathbf{S}^*_{u,v}(t,5)\mathbf{1}$, $u = 0, 1$, $v = 0, 1, 2$, by (3.10):

$$\mathbf{S}^*_{0,0}(t,5)\mathbf{1} = (1.6, -0.4)', \quad \mathbf{S}^*_{0,1}(t,5) \cdot \mathbf{1} = (-1.6, 0.4)', \quad \mathbf{S}^*_{0,2}(t,5) \cdot \mathbf{1} = (0,0)'$$

$$\mathbf{S}^*_{1,0}(t,5)\mathbf{1} = 0, \qquad \mathbf{S}^*_{1,1}(t,5)\mathbf{1} = 0, \qquad \mathbf{S}^*_{1,2}(t,5)\mathbf{1} = 0.$$

Then, substituting in (3.21), we get

$$\mathbf{S}^*(t,5) = \left[\begin{array}{ccc|cccc|cc} .1866 & 1.12 & .2934 & -.225 & .0684 & -1.1506 & -.2928 & -.1432 & .1432 \\ .1866 & 1.12 & .2934 & -.225 & .0684 & -1.1506 & -.2928 & -.1432 & .1432 \\ \hline -.322 & .6 & -.278 & -.4107 & .4853 & .36 & -.4347 & -.5469 & .5469 \end{array}\right].$$

Similarly, we evaluate the remaining subsequences:

$$\mathbf{S}^*(t,0) = \left[\begin{array}{ccc|cccc|cc} .1714 & .8 & .0286 & -.4107 & .4853 & .36 & -.4347 & -1.0137 & .0137 \\ .1714 & .8 & .0286 & -1.2241 & .128 & -.064 & -.84 & -.0801 & 1.0801 \\ \hline -.1094 & 1 & .1094 & .019 & .1756 & -1.0234 & -.1712 & -.1432 & .1432 \end{array}\right],$$

$$\mathbf{S}^*(t,1) = \left[\begin{array}{ccc|cccc|cc} -.6028 & .8 & -.1973 & .4257 & .3543 & -.8114 & .0315 & -.1432 & .1432 \\ .384 & 1.2 & .416 & .4257 & .3543 & -.8114 & .0315 & -1.0768 & -.9232 \\ \hline .1714 & .8 & .0286 & -.6547 & .3781 & .2328 & -.5562 & -.7337 & .3337 \end{array}\right],$$

$$\mathbf{S}^*(t,2) = \left[\begin{array}{ccc|cccc|cc} .1714 & .8 & .0286 & -1.0614 & .1995 & .0208 & -.7589 & -.2669 & .8669 \\ -.8154 & .4 & -.5846 & -.248 & .5568 & .4448 & -.3536 & -.2669 & .8669 \\ \hline -.3067 & .92 & -.0133 & -.4257 & .3543 & -.8114 & .0315 & -.4233 & -.1767 \end{array}\right],$$

$$\mathbf{S}^*(t,3) = \left[\begin{array}{ccc|cccc|cc} .1866 & 1.12 & .2934 & .4257 & .3543 & -.8114 & .0315 & -.8901 & -.71 \\ .1866 & 1.12 & .2934 & -.3877 & -.0031 & -1.2354 & -.3739 & .0435 & .3565 \\ \hline -.1247 & .68 & -.1553 & -.8174 & .3067 & .148 & -.6373 & -.2669 & .8669 \end{array}\right],$$

$$\mathbf{S}^*(t,4) = \left[\begin{array}{ccc|cccc|cc} -.618 & .48 & -.462 & -.4107 & .4853 & .36 & -.4347 & -.2669 & .8669 \\ .3687 & .88 & .1513 & -.4107 & .4853 & .36 & -.4347 & -1.2005 & -.1995 \\ \hline .1866 & 1.12 & .2934 & .1817 & .2471 & -.9386 & -.0901 & -.61 & -.39 \end{array}\right].$$

REFERENCES

Bartholomew, D. J. 1967. *Stochastic Models for Social Processes*, 1st ed., Wiley, Chichester.

Bartholomew, D. J. 1973. *Stochastic Models for Social Processes*, 2nd ed., Wiley, Chichester.

Bartholomew, D. J. 1982. *Stochastic Models for Social Processes*, 3rd ed., Wiley, Chichester.

Bowerman, B. 1974. Nonstationary Markov Decision Processes and Related Topics in Nonstationary Markov Chains. Ph.D. Thesis, Iowa State Univ., Ames.

Bowerman, B., David, H. T., and Isaacson, D. 1977. The convergence of Cesaro averages for certain nonstationary Markov chains, *Stochastic Process. Appl.* 5:221–230.

Huang, C., Isaacson, D., and Vinograde, B. 1976. The rate of convergence of certain non-homogeneous Markov chains, *Z. Wahrsch. Verw. Geb.* 35:141–146.

Iosifescu, M. 1979. *Finite Markov Processes and Their Applications*, Wiley, New York.

Isaacson, D. L., and Madsen, R. W. 1976. *Markov Chains: Theory and Applications*, Wiley, New York.

Pollard, J. H. 1973. *Mathematical Models for the Growth of Human Populations*, Cambridge Univ. P., Cambridge.

Stark, H. M. 1970. *An Introduction to Number Theory*, MIT Press, Cambridge, Mass.

Tsaklidis, G. and Vassiliou, P.-C. G. 1988). *V*-matrices as covariance matrices in homogeneous Markov systems, in *Applications of Matrix Theory* (M. J. C. Gover and S. Barnett, Eds.), Oxford Science Publications, 1989.

Tsaklidis, G. and Vassiliou, P.-C. G., 1990. Infinite products of matrices with some negative elements and row sums equal to one, *Linear Algebra Appl.* 127:41–58.

Vassiliou, P.-C. G. and Gerontidis, I. 1985. Variances and covariance of the grade sizes in manpower systems, *J. Appl. Probab.* 22:583–597.

Wynn, H. P. 1973. Limiting second moments for transient states of Markov chains, *J. Appl. Probab.* 10:891–894.

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